4) Use Green's theorem to evaluate the line integral

$$\oint_c \left(x+y^2\right)dx+\left(1+x^2\right)dy$$

Solution :

$$\oint_c F_1\,dx+F_2\,dy=\iint_R \left(\frac{dF_2}{dx}-¿\,\frac{dF_1}{dy}\right)dx\,dy¿$$

Limits :

$0 \le x \le 1$

$x^3 \le y \le \sqrt{x}$

$$=\int_{x^3}^{\sqrt{x}}\int \left(2x-2y\right)dy\,dx$$

$$=\int_0^1 \left((2x)(\sqrt{x})-\sqrt{x}^2\right)-\left((2x)\left(x^3\right)-x^6\right)dx$$

$$=\int_0^1 \left(2x^{\frac{3}{2}}-x-2x^4+x^6\right)dx$$

$$=\left(\frac{4}{5}x^{\frac{5}{2}}-\frac{x^2}{2}-\frac{2x^5}{5}+\frac{x^7}{7}\right)$$

Substitute 0 and 1

$$=\left(\frac{4}{5}\left(1^{\frac{5}{2}}\right)-\frac{1^2}{2}-\frac{2(1^5)}{5}+\frac{1^7}{7}\right)$$

$$=\frac{3}{70}$$